Rounding real numbers to integers

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Given real numbers $x_1, \ldots, x_n$, we may wish to replace them by integers $a_1, \ldots, a_n$ so that the rounding errors will be small. Clearly, we can make each error $|x_i - a_i| \leq \frac{1}{2}$, but what if we also care about errors of the form $|\sum_{i \in S} x_i - \sum_{i \in S} a_i|$ for certain subsets $S \subseteq \{1, \ldots, n\}$? If we care about all subsets S, then no matter how we choose the integers $a_1, \ldots, a_n$ we may create a large error ($\sim \frac{n}{4}$) for some S. An interesting question is whether we can do well for certain natural families of subsets. The project will focus on some such (solved and unsolved) questions, for example:

- Known: Given $x_1, \ldots, x_n$, it is possible to choose integers $a_1, \ldots, a_n$ so that for every interval I (i.e., $I = \{k, k+1, \ldots, l\}$ for some $1 \leq k \leq l \leq n$) we'll have $|\sum_{i \in I} x_i - \sum_{i \in I} a_i| \leq 1 - \frac{1}{n+1}$.
- Known: Given $x_1, \ldots, x_n$ and a parameter $d \in \{1, 2, \ldots, \lfloor \frac{n+2}{2} \rfloor\}$, it is possible to choose integers $a_1, \ldots, a_n$ so that for every S which is the union of at most d intervals we'll have $|\sum_{i \in S} x_i - \sum_{i \in S} a_i| \leq (1 - \frac{d}{n+1})d$.
- Unknown: Given $x_1, \ldots, x_n$, is it possible to choose integers $a_1, \ldots, a_n$ so that for all $d \in \{1, 2, \ldots, \lfloor \frac{n+2}{2} \rfloor\}$ and every S which is the union of d intervals we'll have $|\sum_{i \in S} x_i - \sum_{i \in S} a_i| \leq (1 - \frac{d}{n+1})d$?
  This question appeared in a 2005 paper, but remains open. One reason to believe that the answer is positive is that if we care only about $d = 1, 2$ then this is feasible. No proof or counterexample is known for $d = 1, 2, 3$.

The methods are elementary (of combinatorial nature) but require clever ideas.