Random Walks with Self-Interactions

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A random walk on the integers $\mathbb{Z}$ is a random sequence $\{X_n\}_{n=0}^{\infty}$ which satisfies $X_0 = z_0 \in \mathbb{Z}$ and

$$P(X_{n+1} = z+1 | X_n = z) = p, \quad P(X_{n+1} = z-1 | X_n = z) = 1-p,$$

where $p \in (0,1)$ is a parameter. In words, at each time $n = 1, 2, \cdots$, the random walk jumps either one step to the right or one step to the left, with respective probabilities p and $1-p$. If $p = \frac{1}{2}$, then the random walk is called a symmetric random walk. If $p > \frac{1}{2}$, then with probability one the random walk runs off to $+\infty$: $P(\lim_{n\to\infty} X_n = +\infty) = 1$, while if $p < \frac{1}{2}$, then with probability one the random walk runs off to $-\infty$: $P(\lim_{n\to\infty} X_n = -\infty) = 1$. On the other hand, if $p = \frac{1}{2}$, the random walk visits every site in $\mathbb{Z}$ infinitely often:

$$(1) \qquad P(\limsup_{n\to\infty} X_n = +\infty \text{ and } \liminf_{n\to\infty} X_n = -\infty) = 1.$$

A random walk that satisfies (1) is called recurrent, while a random walk that satisfies $P(\lim_{n\to\infty} |X_n| = \infty) = 1$ is called transient. Thus, the above random walk is recurrent if $p = \frac{1}{2}$ and transient if $p \neq \frac{1}{2}$.

Consider now the following "random walk with self-interactions." The random walk has four parameters associated with it: $p, q \in (0,1)$ and $R, L \in \mathbb{N}$. Initially each site $z \in \mathbb{Z}$ is set to either the p-mode or the q-mode. From a site in the p-mode the walk jumps right with probability p and left with probability $1-p$, whereas from a site in the q-mode these probabilities are q and $1-q$, respectively. Also, a site z switches from the q-mode to the p-mode after the walk jumps right from z on R consecutive visits to z, and a site z switches from the p-mode to the q-mode after the walk jumps left from z on L consecutive visits to z. Convince yourself that it makes sense to call the case $p > q$ "positive feedback" and to call the case $p < q$ "negative feedback."

We are interested in the question of transience or recurrence of this random walk. Let

$$(2) \qquad \alpha = \frac{p \cdot \left[(1-q)q^R(1-(1-p)^L)\right] \;+\; q \cdot \left[p(1-p)^L(1-q^R)\right]}{\left[(1-q)q^R(1-(1-p)^L)\right] \;+\; \left[p(1-p)^L(1-q^R)\right]}.$$

It turns out that if $\alpha \neq \frac{1}{2}$, then the process is transient. More specifically, if $\alpha > \frac{1}{2}$, then $P(\lim_{n\to\infty} X_n = +\infty) = 1$, while if $\alpha < \frac{1}{2}$, then $P(\lim_{n\to\infty} X_n = -\infty) = 1$. When $\alpha = \frac{1}{2}$, the transience/recurrence behavior usually depends on the initial "environment." We call the case $\alpha = \frac{1}{2}$ the critical case, and the case $\alpha \neq \frac{1}{2}$ the non-critical case. At each site $z \in \mathbb{Z}$, we attach an initial environment of the form (p, i) or (q, j), with $0 \leq i \leq L - 1$ and $0 \leq j \leq R - 1$. The first coordinate–p or q–tells us which mode the site starts out in. The second component tells us how many "strikes against this mode" we begin with. Thus, for example, if $R = 3$ and the initial environment at some particular site z is $(q, 1)$, then the first time the process gets to site z, it jumps in the q-mode. If it jumps left from z, then the next time it is at z, the site will be in the environment $(q, 0)$, while if it jumped right from z, then the next time it is at z, the site will be in the environment $(q, 2)$. Let's say that this latter scenario occurred. Then the next time the process reaches site z, it sees the environment $(q, 2)$. So it jumps using the q-mode. If it jumps left from z, then the next time it is at site z, the site will be in the environment $(q, 0)$, while if it jumps right from z, then the next time it is at site z, the site will be in the environment $(p, 0)$.

There are two aspects of this project. One aspect is to do some rather complicated calculations to determine whether the process is recurrent or transient in the critical case with $R = L = 3$. (The case $R = L = 2$ was treated in a paper I wrote with a post-doctoral fellow of mine.) The other aspect of the project is to change the underlying interaction and determine the correct formula for α in this new situation. Here is an example of changing the underlying interaction. Switch from the p mode to the q mode at a site z whenever three of the last five visits to z resulted in jumping left, and switch from the q mode to the p mode whenever four of the last five visits to z resulted in jumping right.

In preparation for this project, I will send the students some material that they should read before they arrive. Also, the first day of the project will be devoted mainly to my lecturing and giving background material.